

General Formulation of Quantum Analysis

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Abstract

A general formulation of noncommutative or quantum derivatives for operators in a Banach space is given on the basis of the Leibniz rule, irrespective of their explicit representations such as the Gâteaux derivative or commutators. This yields a unified formulation of quantum analysis, namely the invariance of quantum derivatives, which are expressed by multiple integrals of ordinary higher derivatives with hyperoperator variables. Multivariate quantum analysis is also formulated in the present unified scheme by introducing a partial inner derivation and a rearrangement formula. Operator Taylor expansion formulas are also given by introducing the two hyperoperators $\delta_{A \rightarrow B} \equiv -\delta_A^{-1} \delta_B$ and $d_{A \rightarrow B} \equiv \delta_{(-\delta_A^{-1} B); A}$ with the inner derivation $\delta_A : Q \mapsto [A, Q] \equiv AQ - QA$. Physically the present noncommutative derivatives express quantum fluctuations and responses.

I. Introduction

Recently noncommutative calculus has attracted the interest of many mathematicians and physicists^{1–15}. The present author^{10–15} has introduced the quantum derivative $df(A)/dA$ of the operator function $f(A)$ in the Gâteaux differential^{1–3}

$$df(A) = \lim_{h \rightarrow 0} \frac{f(A + h dA) - f(A)}{h} \equiv \frac{df(A)}{dA} \cdot dA. \quad (1.1)$$

Here the quantum derivative $df(A)/dA$ is a hyperoperator^{10–15}, which maps an arbitrary operator dA to the differential $df(A)$ in a Banach space. There is also an algebraic definition^{8,9,12,13} of the differential $df(A)$ as

$$df(A) = [H, f(A)] \quad (1.2)$$

for an auxiliary operator H in a Banach space. This differential depends on H . In particular, we have

$$dA = [H, A]. \quad (1.3)$$

The property that $d^2A = 0$ requires the following condition

$$[H, [H, A]] = [H, dA] = 0. \quad (1.4)$$

In the previous papers^{10–13}, we have shown that the differential $df(A)/dA$ defined in (1.1) is expressed by

$$\frac{df(A)}{dA} = \frac{\delta_{f(A)}}{\delta_A}, \quad (1.5)$$

where δ_A denotes an inner derivation defined by

$$\delta_A Q = [A, Q] = AQ - QA \quad (1.6)$$

for an arbitrary operator Q in a Banach space. The ratio of the two hyperoperators in (1.5) is well defined^{10–13} when $f(A)$ is a convergent operator power series.

On the other hand, the derivative $df(A)/dA$ defined through the commutator (1.2) is also expressed^{9,12,13,16} by Eq.(1.5). This is easily derived as follows. From Eq.(1.2), we have

$$\begin{aligned} df(A) &= \delta_H f(A) = -\delta_{f(A)} H = -\delta_A^{-1} \delta_A \delta_{f(A)} H \\ &= \delta_A^{-1} \delta_{f(A)} (-\delta_A H) = \delta_A^{-1} \delta_{f(A)} [H, A] = \frac{\delta_{f(A)}}{\delta_A} dA, \end{aligned} \quad (1.7)$$

using the commutativity of δ_A and $\delta_{f(A)}$. The meaning of the formal inverse δ_A^{-1} in Eq.(1.7) will be discussed in the succeeding section. The above results suggest

that the quantum derivative $df(A)/dA$ defined in Eq.(1.8) is invariant for any choice of definitions of the differential $df(A)$. One of the main purposes of the present paper is to make a unified formulation of quantum analysis and to prove the invariance of the quantum derivative $df(A)/dA$ defined in

$$df(A) \equiv \frac{df(A)}{dA} \cdot dA \quad (1.8)$$

for any differential $df(A)$ satisfying the Leibniz rule

$$d(fg) = (df)g + f(dg). \quad (1.9)$$

In Sec.II, some mathematical preparations are made on the formal inverse δ_A^{-1} of the inner derivation δ_A . In Sec.III we present a general formulation of quantum derivatives using the hyperoperators $\delta_A^{-1}\delta_{dA}$ and $\delta_{(-\delta_A^{-1}dA);A}$. Theorem I states the invariance of the differential $df(A)$ for any choice of definitions of $df(A)$. Theorem II gives the invariance of the derivative, $df(A)/dA$. Theorem III presents algebraic expressions of higher differentials $\{d^n f(A)\}$. Theorem IV gives multiple integral representations of higher derivatives $\{d^n f(A)/dA^n\}$. Theorem V presents a general Taylor expansion formula of $f(A + xB)$ in terms of higher derivatives $\{d^n f(A)/dA^n\}$ for the noncommutative operators A and B . A shift-hyperoperator $\mathcal{S}_A(B) : f(A) \mapsto f(A+B)$ is also formulated. A general formulation of multivariate quantum analysis is given in Sec.IV, by introducing a partial inner derivation and a rearrangement formula. In Sec.V, an auxiliary operator method is briefly discussed, and it is extended to multivariate operator functions. In Sec.VI, some general remarks and applications to exponential product formulas are briefly mentioned. Summary and discussion are given in Sec.VII.

II. Inner Derivation, its Formal Inverse and Uniqueness

In the present section, we introduce the two hyperoperators $(-\delta_A^{-1}\delta_B)$ and $\delta_{(-\delta_A^{-1}B)}$, and discuss the existence and uniqueness of these hyperoperators in the

domain \mathcal{D}_A , which is defined by the set of convergent power series of the operator A in a Banach space. In general, it seems to be meaningless to use the symbol δ_A^{-1} , because the inverse of the inner derivation δ_A does not necessarily exist and furthermore is not unique even if it exists. Fortunately in our problem, only the combinations $(-\delta_A^{-1}\delta_B)$ and $\delta_{(-\delta_A^{-1}B)}$ appear in our quantum analysis of single-variable functions. Thus there is a possibility to define them uniquely.

(i) Hyperoperator $\delta_{A \rightarrow B} \equiv -\delta_A^{-1}\delta_B$

First we show that the hyperoperator $(-\delta_A^{-1}\delta_B)$ is well defined when it operates on a function $f(A)$ in the domain \mathcal{D}_A . For this purpose, we confirm that

$$\delta_{A \rightarrow B}A \equiv (-\delta_A^{-1}\delta_B)A = \delta_A^{-1}(-\delta_B A) = \delta_A^{-1}\delta_A B = B, \quad (2.1)$$

namely $\delta_{A \rightarrow B} : A \mapsto B$. More generally, we have

$$\delta_{A \rightarrow B}A^n = \sum_{k=0}^{n-1} A^k (\delta_{A \rightarrow B}A) A^{n-k-1} = \sum_{k=0}^{n-1} A^k B A^{n-k-1} \quad (2.2)$$

for any positive integer n . Thus, the hyperoperator $\delta_{A \rightarrow B} \equiv -\delta_A^{-1}\delta_B$ is well defined, at least, in the domain \mathcal{D}_A . Thus, the existence of $\delta_{A \rightarrow B}$ has been shown, but it is not unique. In fact, we put

$$\delta_{A \rightarrow B}f(A) = F(A, B) \quad (2.3)$$

which is constructed by the above procedure. Then, $F(A, B) + G(A)$ may be also a solution of $(-\delta_A^{-1}\delta_B)f(A)$, because

$$-\delta_B f(A) = \delta_A F(A, B) + \delta_A G(A) \quad (2.4)$$

for any operator $G(A)$ in a Banach space. If we impose, besides the Leibniz rule, the linearity of the hyperoperator $\delta_{A \rightarrow B}$, namely

$$\delta_{A \rightarrow B}(f(A) + g(A)) = \delta_{A \rightarrow B}f(A) + \delta_{A \rightarrow B}g(A), \quad (2.5)$$

and

$$\delta_{A \rightarrow B}(af(A)) = a\delta_{A \rightarrow B}f(A) \quad (2.6)$$

for a complex number a , then the uniqueness of $\delta_{A \rightarrow B}$ is assured. In fact, the expression $F(A, B)$ in (2.3) is obtained explicitly by using this linearity of the hyperoperator $\delta_{A \rightarrow B}$.

In order to study the role of the hyperoperator $\delta_{A \rightarrow B}$ more explicitly, we introduce the symmetrized product $\{A^m B^n\}_{\text{sym}(A, B)}$ by

$$\{A^m B^n\}_{\text{sym}(A, B)} \equiv \sum_{k_1 + \dots + k_{n+1} = m, k_j \geq 0} A^{k_1} B A^{k_2} \dots A^{k_n} B A^{k_{n+1}}, \quad (2.7)$$

where $m, n, \{k_j\}$ denote non-negative integers. This symmetrized product is also written as

$$\{A^m B^n\}_{\text{sym}(A, B)} = \frac{1}{n!} \left[\frac{d^n}{dx^n} (A + xB)^{m+n} \right]_{x=0}. \quad (2.8)$$

Then, Eq.(2.2) is expressed by

$$\delta_{A \rightarrow B} A^m = \{A^{m-1} B\}_{\text{sym}(A, B)}. \quad (2.9)$$

Hereafter, we write $\{\dots\}_{\text{sym}(A, B)}$ simply as $\{\dots\}_{\text{sym}}$, when no confusion arises. Similarly we obtain

$$\delta_{A \rightarrow B} \{A^m B\}_{\text{sym}} = \{A^{m-1} B^2\}_{\text{sym}}, \quad (2.10)$$

because

$$\begin{aligned} -\delta_B \{A^m B\}_{\text{sym}} &= -\delta_B \left(\sum_{k=0}^m A^k B A^{m-k} \right) \\ &= \sum_{k=0}^m [(-\delta_B A^k) B A^{m-k} + A^k B (-\delta_B A^{m-k})] \end{aligned} \quad (2.11)$$

using the Leibniz rule. Using the commutativity of A and δ_A and the relation (2.9), namely $-\delta_B A^m = \delta_A \{A^{m-1} B\}_{\text{sym}}$, we have

$$\begin{aligned} -\delta_B \{A^m B\}_{\text{sym}} &= \sum_{k=1}^m (\delta_A \{A^{k-1} B\}_{\text{sym}}) B A^{m-k} + \sum_{k=0}^{m-1} A^k B (\delta_A \{A^{m-k-1} B\}_{\text{sym}}) \\ &= \delta_A \{A^{m-1} B^2\}_{\text{sym}}. \end{aligned} \quad (2.12)$$

In general, we have the following formula.

Formula 1 : For non-negative integers $m(\geq 1)$ and n and for any operators A and B in a Banach space, we have

$$-\delta_B\{A^m B^n\}_{\text{sym}} = \delta_A\{A^{m-1} B^{n+1}\}_{\text{sym}} \quad (2.13)$$

namely

$$\delta_{A \rightarrow B}\{A^m B^n\}_{\text{sym}} = \{A^{m-1} B^{n+1}\}_{\text{sym}} \quad (2.14)$$

Consequently, the domain of the hyperoperator $\delta_{A \rightarrow B}$ is extended to the region $\mathcal{D}_{\text{sym}(A,B)}$ which is a set of convergent noncommuting symmetrized power series of A and B .

The proof of this formula is given as follows. First note that

$$\delta_{A+xB}(A+xB)^{m+n} = 0, \quad (2.15)$$

namely

$$-x\delta_B(A+xB)^{m+n} = \delta_A(A+xB)^{m+n}. \quad (2.16)$$

By comparing the $(n+1)$ -th terms of the both sides of (2.16) in x and using the relation (2.8), we obtain Eq.(2.13) and consequently Eq.(2.14). An alternative derivation of Eq.(2.13) will be given by extending the procedure shown in Eqs.(2.11) and (2.12).

Next we study the property of the power hyperoperators $\{\delta_{A \rightarrow B}^k\}$. It is easy to show the following formula.

Formula 2 : For non-negative integers $k, m(\geq k)$ and n and for any operators in a Banach space, we have

$$\delta_{A \rightarrow B}^k\{A^m B^n\}_{\text{sym}} = \{A^{m-k} B^{n+k}\}_{\text{sym}}, \quad (2.17a)$$

and

$$\delta_{A \rightarrow B}^k\{A^m B^n\}_{\text{sym}} = 0 \quad \text{if} \quad m < k. \quad (2.17b)$$

This gives the following general formula.

Formula 3 : When $f(A)$ is a convergent operator power series of an operator A in a Banach space, we have

$$\begin{aligned}\delta_{A \rightarrow B}^n \{f(A)B^m\}_{\text{sym}} &= \int_0^1 dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \{f^{(n)}(t_n A)B^{m+n}\}_{\text{sym}} \\ &= \frac{1}{(n-1)!} \int_0^1 dt (1-t)^{n-1} \{f^{(n)}(tA)B^{m+n}\}_{\text{sym}}.\end{aligned}\quad (2.18)$$

Here, $f^{(n)}(x)$ denotes the n th derivative of $f(x)$.

(ii) Hyperoperators $\delta_{(-\delta_A^{-1}B)}$ and $d_{A \rightarrow B} \equiv \delta_{(-\delta_A^{-1}B);A}$

An operator H defined by

$$-\delta_A H = B \quad (2.19)$$

does not necessarily exist, as is well known. However, the hyperoperator $\delta_{(-\delta_A^{-1}B)}$ is well defined, at least, when it operates on $f(A)$ in the domain \mathcal{D}_A for an operator A in a Banach space. In fact, we can interpret it as

$$\begin{aligned}\delta_{(-\delta_A^{-1}B)} A^m &= \sum_{k=0}^{m-1} A^k (\delta_{(-\delta_A^{-1}B)} A) A^{m-k-1} = \sum_{k=0}^{m-1} A^k (\delta_A \delta_A^{-1} B) A^{m-k-1} \\ &= \sum_{k=0}^{m-1} A^k B A^{m-k-1} = \delta_{A \rightarrow B} A^m.\end{aligned}\quad (2.20)$$

In other words, the formal hyperoperator δ_A^{-1} in $\delta_{(-\delta_A^{-1}B)}$ should be interpreted as a hyperoperator operating on the left-hand-side hyperoperator δ_A (not on the right-hand-side operator B). In this interpretation, the hyperoperator $\delta_{(-\delta_A^{-1}B)}$ is defined even when the operator $(-\delta_A^{-1}B)$ does not exist.

In general, we obtain the following formula.

Formula 4: Under the requirement of the linearity of the hyperoperators $\delta_{A \rightarrow B}$ and $\delta_{(-\delta_A^{-1}B)}$, we have

$$\delta_{(-\delta_A^{-1}B)} f(A) = \delta_{A \rightarrow B} f(A) = \int_0^1 \{f^{(1)}(tA)B\}_{\text{sym}} dt \quad (2.21)$$

for any operator $f(A) \in \mathcal{D}_A$. Here, $f^{(1)}(x)$ denotes the first derivative of $f(x)$.

It should be remarked that the hyperoperator $\delta_{(-\delta_A^{-1}dA)}$ is a kind of differential defined only in the domain \mathcal{D}_A , whereas the hyperoperator $\delta_{A \rightarrow dA}$ is defined in a wider domain but is not a differential in the domain $\mathcal{D}_{\text{sym}(A, dA)}$ outside of the domain \mathcal{D}_A .

As was discussed before, the operator $H \equiv -\delta_A^{-1}B$ does not necessarily exist, and it is difficult to define the power hyperoperators $\{\delta_{(-\delta_A^{-1}B)}^n\}$ for $n \geq 2$ when $H \equiv -\delta_A^{-1}B$ does not exist. Furthermore, they are complicated¹² even if they do exist, unless H commutes with B . Thus, we define the following partial inner derivation

$$d_{A \rightarrow B} \equiv \delta_{(-\delta_A^{-1}B); A}, \quad (2.22)$$

by which the commutator $\delta_{(-\delta_A^{-1}B)}$ is taken only with the operator A in a multivariate operator $f(A, B)$. For example, we have

$$d_{A \rightarrow B}(ABA) = (\delta_{(-\delta_A^{-1}B)}A)BA + AB(\delta_{(-\delta_A^{-1}B)}A). \quad (2.23)$$

This new hyperoperator $d_{A \rightarrow B}$ is defined in the domain $\mathcal{D}_{A, B}$ which is a set of convergent noncommuting power series of the operators A and B . Clearly, $d_{A \rightarrow B}$ is a kind of differential satisfying the Leibniz rule for $B = dA$.

Next we study the power hyperoperators $\{d_{A \rightarrow B}^n\}$. Clearly they are also differentials defined in the domain $\mathcal{D}_{A, B}$. It will be interesting to find the relation between $d_{A \rightarrow B}^n$ and $\delta_{A \rightarrow B}^n$.

First note that

$$\begin{aligned} d_{A \rightarrow B}^2 A^m &= d_{A \rightarrow B} \{A^{m-1}B\}_{\text{sym}} = d_{A \rightarrow B} \left(\sum_{j=0}^{m-1} A^j B A^{m-j-1} \right) \\ &= \sum_{j=1}^{m-1} \sum_{k=0}^{j-1} A^{j-k-1} B A^k B A^{m-j-1} + \sum_{j=0}^{m-2} \sum_{k=0}^{m-j-2} A^j B A^k B A^{m-j-k-2} \\ &= 2\{A^{m-2}B^2\}_{\text{sym}} = 2\delta_{A \rightarrow B}^2 A^m \end{aligned} \quad (2.24)$$

for $m \geq 2$. In general, we obtain the following formula.

Formula 5 : For $m \geq n$ and for any operators A and B in a Banach space, we have

$$d_{A \rightarrow B}^n A^m = n! \delta_{A \rightarrow B}^n A^m. \quad (2.25)$$

We have also $d_{A \rightarrow B}^n A^m = 0$ for $m < n$. More generally, we have

$$d_{A \rightarrow B}^n f(A) = n! \delta_{A \rightarrow B}^n f(A), \quad (2.26)$$

when $f(A) \in \mathcal{D}_A$.

The proof of Formula 5 is given by mathematical induction using the following lemma and Formula 2.

Lemma 1 : For non-negative integers $m(\geq 1)$ and n and for any operators A and B in a Banach space, we have

$$d_{A \rightarrow B}\{A^m B^n\}_{\text{sym}} = (n+1)\{A^{m-1} B^{n+1}\}_{\text{sym}}. \quad (2.27)$$

This is easily proved by using the definition (2.7) of $\{A^m B^n\}_{\text{sym}}$ as in Eq.(2.24). Formula 5 can be also confirmed directly from the consideration on the number of permutations of B^n . More intuitively, $\delta_{A \rightarrow dA}^n$ denotes an ordered partial differential¹¹. On the other hand, $d_{A \rightarrow dA}^n$ denotes the n th differential, as will be discussed later. Consequently we have Formula 5.

It should be remarked here that the hyperoperator $d_{A \rightarrow B}^n$ is equivalent to $\delta_{(-\delta_A^{-1}B)}^n$ when $H \equiv -\delta_A^{-1}B$ exists and it commutes with B . This equivalence has been already used implicitly in the previous papers^{12,13}.

With these preparations, we discuss a general theory of derivatives of $f(A)$ with respect to the operator A itself in the succeeding section.

III. Quantum Derivative, its Invariance and Operator Taylor Expansion

In the present section, we give a general formulation of quantum derivatives $\{d^n f(A)/dA^n\}$ which do not depend on the definition of the differential $df(A)$. Our starting point of this general theory is that the differential hyperoperator “ d ” satisfies the Leibniz rule (1.9) and that it is a linear hyperoperator.

(i) Quantum derivative and its invariance

Now we start with the following identity

$$Af(A) = f(A)A, \quad (3.1)$$

when $f(A) \in \mathcal{D}_A$. Then, we have

$$d(Af(A)) = d(f(A)A), \quad (3.2)$$

which is rewritten as

$$(dA)f(A) + Adf(A) = (df(A))A + f(A)dA, \quad (3.3)$$

using the Leibniz rule. This is rearranged as follows :

$$Adf(A) - (df(A))A = f(A)dA - (dA)f(A). \quad (3.4)$$

That is, we have

$$\delta_A df(A) = \delta_{f(A)} dA. \quad (3.5)$$

This is our desired formula on the differential $df(A)$.

In order to discuss the solution of Eq.(3.5), we rewrite Eq.(3.5) as

$$\delta_A df(A) = -\delta_{dA} f(A). \quad (3.6)$$

Obviously, $df(A)$ has a linearity property with respect to $f(A)$. Thus, the solution $df(A)$ of Eq.(3.6) is uniquely given in the form

$$df(A) = \delta_{A \rightarrow dA} f(A), \quad (3.7)$$

using the hyperoperator $\delta_{A \rightarrow dA} \equiv -\delta_A^{-1} \delta_{dA}$ introduced in Section II. This is also rewritten as

$$\begin{aligned} df(A) &= \int_0^1 dt \{f^{(1)}(tA) dA\}_{\text{sym}(A, dA)} \\ &= \int_0^1 f^{(1)}(A - t\delta_A) dt \cdot dA, \end{aligned} \quad (3.8)$$

using Formula 4, namely Eq.(2.21).

The second equality of Eq.(3.8) is proven as follows. First we prove it when $f(A) = A^m$ for an arbitrary positive integer m . Clearly we have

$$\begin{aligned} dA^m &= \int_0^1 dt \{f^{(1)}(tA) dA\}_{\text{sym}(A, dA)} \\ &= (m \int_0^1 t^{m-1} dt) \{A^{m-1} dA\}_{\text{sym}(A, dA)} \\ &= \{A^{m-1} dA\}_{\text{sym}}. \end{aligned} \quad (3.9)$$

On the other hand, we obtain

$$\begin{aligned} \int_0^1 f^{(1)}(A - t\delta_A) dt \cdot dA &= \int_0^1 f^{(1)}((1-t)A + t(A - \delta_A)) dt \cdot dA \\ &= m \sum_{k=0}^{m-1} {}_{m-1}C_k \int_0^1 (1-t)^k t^{m-1-k} dt A^k (A - \delta_A)^{m-1-k} \cdot dA \\ &= m \sum_{k=0}^{m-1} {}_{m-1}C_k B(k+1, m-k) A^k (A - \delta_A)^{m-1-k} \cdot dA \\ &= \sum_{k=0}^{m-1} A^k (A - \delta_A)^{m-1-k} \cdot dA \\ &= \sum_{k=0}^{m-1} A^k (dA) A^{m-1-k} = \{A^{m-1} dA\}_{\text{sym}}, \end{aligned} \quad (3.10)$$

using the beta function $B(x, y)$, the binomial coefficient ${}_m C_k$, the commutativity of A and δ_A , and the following relation¹⁰

$$(A - \delta_A)^n \cdot dA = (dA) \cdot A^n. \quad (3.11)$$

Thus, the second equality of Eq.(3.8) holds for $f(A) \in \mathcal{D}_A$.

Furthermore we can derive the following relation.

Lemma 2 : When $f(A) \in \mathcal{D}_A$, we have

$$\delta_{f(A)} = f(A) - f(A - \delta_A). \quad (3.12)$$

Using this lemma, we obtain

$$\begin{aligned} \delta_{f(A)} &= f(A) - f(A - \delta_A) \\ &= \delta_A(f^{(1)}(A) - \frac{1}{2}\delta_A f^{(2)}(A) + \dots + \frac{(-1)^{n-1}}{n!}\delta_A^{n-1}f^{(n)}(A) + \dots) \\ &= \delta_A \int_0^1 f^{(1)}(A - t\delta_A)dt. \end{aligned} \quad (3.13)$$

This is formally written as

$$\int_0^1 f^{(1)}(A - t\delta_A)dt = \delta_A^{-1}\delta_{f(A)} = \frac{\delta_{f(A)}}{\delta_A}. \quad (3.14)$$

Thus, summarizing Eqs.(3.5),(3.7),(3.8) and (3.14), we obtain the following theorem on the differential $df(A)$.

Theorem I : When $f(A) \in \mathcal{D}_A$, we have

$$\delta_A df(A) = \delta_{f(A)} dA, \quad (3.15)$$

and consequently

$$\begin{aligned} df(A) &= \delta_{A \rightarrow dA} f(A) \\ &= \int_0^1 f^{(1)}(A - t\delta_A)dt \cdot dA = \frac{\delta_{f(A)}}{\delta_A} \cdot dA \end{aligned} \quad (3.16)$$

for any choice of definitions of the differential $df(A)$.

It should be noted that the ratio of the two hyperoperators $\delta_{f(A)}$ and δ_A is well defined for $f(A) \in \mathcal{D}_A$, as was discussed in the preceding section. We define¹⁰⁻¹² the quantum derivative $df(A)/dA$ in Eq.(1.8), namely

$$df(A) = \frac{df(A)}{dA} \cdot dA. \quad (3.17)$$

That is, the derivative $df(A)/dA$ is a hyperoperator which maps an arbitrary operator dA to the differential $df(A)$ given by Eq.(3.16). Thus, we arrive at the following invariance theorem on the quantum derivative defined in Eq.(3.17).

Theorem II (Invariance of Quantum Derivative) : *When $f(A) \in \mathcal{D}_A$, the quantum derivative $df(A)/dA$ is invariant for any choice of definitions of the differential $df(A)$ satisfying the Leibniz rule, and it is given by*

$$\frac{df(A)}{dA} = \frac{\delta_{f(A)}}{\delta_A} = \int_0^1 f^{(1)}(A - t\delta_A) dt. \quad (3.18)$$

Clearly, the ratio of the two hyperoperators $\delta_{f(A)}$ and δ_A does not depend on the choice of definitions of the differential $df(A)$. This invariance has been also discussed by Nazaikinskii et al⁹. in a different formulation based on the Feynman index method. The present confirmation of the invariance is more direct and transparent.

(ii) Higher derivatives and operator Taylor expansion

Now we discuss higher-order differentials $\{d^n f(A)\}$ and higher derivatives $\{d^n f(A)/dA^n\}$.

(ii-1) Higher-order differentials and derivatives

The hyperoperator $d_{A \rightarrow B}$ introduced in Eq.(2.22) is a derivation satisfying the Leibniz rule (1.9). Thus, $d_{A \rightarrow B}$ is a kind of differential hyperoperator, when $B = dA$. We prove here the following theorem.

Theorem III: *The n th differential $d^n f(A)$ is expressed by*

$$d^n f(A) = d_{A \rightarrow dA}^n f(A) \quad (3.19)$$

for any choice of definitions of the differential $df(A)$, when $f(A) \in \mathcal{D}_A$.

The proof is given as follows. First note the following recursive formula¹⁰(3.21)

obtained by differentiating Eq.(3.5), namely

$$\delta_A df(A) = \delta_{f(A)} dA \quad (3.20)$$

repeatedly.

Formula 6 : When $f(A) \in \mathcal{D}_A$, we have

$$\delta_A d^n f(A) = n \delta_{d^{n-1} f(A)} dA = -n \delta_{dA} d^{n-1} f(A). \quad (3.21)$$

This gives the following result :

Formula 7 : When $f(A) \in \mathcal{D}_A$, we have

$$d^n f(A) = n \delta_{A \rightarrow dA} d^{n-1} f(A) = n! \delta_{A \rightarrow dA}^n f(A), \quad (3.22)$$

using the hyperoperator $\delta_{A \rightarrow B} = -\delta_A^{-1} \delta_B$ introduced in Sec.II.

Here we have also used the relation $df(A) = \delta_{A \rightarrow dA} f(A)$ given in Eq.(3.7). Using Formula 5, namely Eq.(2.26), we arrive at Theorem III. This result means that any differential hyperoperator d is generally expressed by

$$d = d_{A \rightarrow dA} \quad (3.23)$$

in the domain $\mathcal{D}_{A, dA}$.

Next we define¹⁰ the higher derivatives $\{d^n f(A)/dA^n\}$ through the relation

$$d^n f(A) = \frac{d^n f(A)}{dA^n} : \underbrace{dA \cdots \cdots dA}_n. \quad (3.24)$$

Here, $d^n f(A)/dA^n$ denotes a hyperoperator which maps a set of the operators $(dA, \dots, dA) \equiv: dA \cdots \cdots dA$ to $d^n f(A)$. In an ordinary mathematical notation, one may prefer to write as

$$d^n f(A) = \frac{d^n f(A)}{dA^n} (\underbrace{dA, \dots, dA}_n). \quad (3.25)$$

However, as was emphasized before¹⁰, the product form (3.24) is essential in the present quantum analysis. That is, we use the product form (3.24) only when the derivative $d^n f(A)/dA^n$ is expressed explicitly in terms of A and the inner derivations $\{\delta_j\}$ defined by¹⁰

$$\delta_j : \underbrace{dA \cdot - \cdot - \cdot dA}_n = (dA)^{j-1}(\delta_A dA)(dA)^{n-j}. \quad (3.26)$$

If we use the notation (3.25), this property of product (3.26) and

$$A : \underbrace{dA \cdot - \cdot - \cdot dA}_n = A(dA)^n \quad (3.27)$$

can not be shown explicitly. Clearly A and $\{\delta_j\}$ are commutable with each other.

(ii-2) Integral representation of $d^n f(A)/dA^n$

Here we express $d^n f(A)/dA^n$ explicitly in an integral form in terms of the above inner derivations $\{\delta_j\}$. Our result is given by the following theorem.

Theorem IV : *When $f(x)$ is analytic and $f(A) \in \mathcal{D}_A$, any higher derivative $d^n f(A)/dA^n$ exists uniquely for any choice of definitions of the differential $df(A)$, and it is given explicitly in the form*

$$\frac{d^n f(A)}{dA^n} = n! \int_0^1 dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n f^{(n)}(A - \sum_{j=1}^n t_j \delta_j). \quad (3.28)$$

Here, $f^{(n)}(x)$ denotes the n th derivative of $f(x)$.

The proof is given as follows. Once the above integral representation (3.28) is derived, the uniqueness of it is clear. In the case of $n = 1$, we have

$$\frac{df(A)}{dA} = \frac{\delta_{f(A)}}{\delta_A} = \frac{f(A) - f(A - \delta_A)}{\delta_A} = \int_0^1 f^{(1)}(A - t\delta_A) dt \quad (3.29)$$

from Theorem II and Lemma 2.

The n th derivative of $f(A)$ divided by $n!$, namely $\hat{f}_n(A, \delta_1, \dots, \delta_n)$ defined by

$$\hat{f}_n(A, \delta_1, \dots, \delta_n) : (dA)^n \equiv \frac{1}{n!} \frac{d^n f(A)}{dA^n} : (dA)^n = \delta_{A \rightarrow dA}^n f(A) \quad (3.30)$$

is shown from Formula 6 to satisfy the following relation

$$(\delta_1 + \cdots + \delta_n) \hat{f}_n(A, \delta_1, \cdots, \delta_n) = \hat{f}_{n-1}(A, \delta_1, \cdots, \delta_{n-1}) - \hat{f}_{n-1}(A - \delta_1, \delta_2, \cdots, \delta_n). \quad (3.31a)$$

When $f(A) = A^k$, Eq.(3.31a) means that

$$\delta_A d^n A^k = n(d^{n-1} A^k dA - d^{n-1}(A - \delta_A)^k dA), \quad (3.31b)$$

which is equivalent by Formula 6 to saying that

$$dA d^{n-1} A^k = d^{n-1}(A - \delta_A)^k dA. \quad (3.31c)$$

The solution of (3.31a) with the condition (3.29) for $n = 1$ is proven to be given by

$$\hat{f}_n(A, \delta_1, \cdots, \delta_n) = \int_0^1 dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n f^{(n)}(A - \sum_{j=1}^n t_j \delta_j), \quad (3.32)$$

using the commutativity of A and $\{\delta_j\}$, and the following formula for $t = 1$.

Formula 8 : For any positive integers m and n , we have

$$\begin{aligned} (x_1 + \cdots + x_n) \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n f^{(m+1)}(tx - \sum_{j=1}^n t_j x_j) \\ = \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-2}} dt_{n-1} \left(f^{(m)}(tx - \sum_{j=1}^{n-1} t_j x_j) \right. \\ \left. - f^{(m)}(t(x - x_1) - \sum_{j=1}^{n-1} t_j x_{j+1}) \right), \end{aligned} \quad (3.33)$$

when $f(x)$ is a convergent power series of x .

Proof of Formula 8 : Let $h_{n,m}(t; x, x_1, \cdots, x_n)$ be the left-hand side of (3.33) minus the right-hand side of (3.33). Then, we have

$$\frac{d}{dt} h_{n,m}(t; x, x_1, \cdots, x_n) = x h_{n,m+1}(t; x, x_1, \cdots, x_n) + h_{n-1,m}(t; x - x_1, x_2, \cdots, x_n). \quad (3.34)$$

If we assume that $h_{n-1,m}(t; x, x_1, \cdots, x_{n-1}) = 0$ for all positive integers m and for any x , and $\{x_j\}$, then we obtain

$$\frac{d}{dt} h_{n,m}(t; x, x_1, \cdots, x_n) = x h_{n,m+1}(t; x, x_1, \cdots, x_n). \quad (3.35)$$

Thus, we derive

$$\frac{d^N}{dt^N} h_{n,m}(t; x, x_1, \dots, x_n) = x h_{n,m+N}(t; x, x_1, \dots, x_n) \quad (3.36)$$

for any positive integer N . Thus, when $f(x)$ is a polynomial of x , we have $h_{n,m+N}(t; x, x_1, \dots, x_n) = 0$ for a large N . Clearly we have

$$\left[\frac{d^k}{dt^k} h_{n,m}(t; x, x_1, \dots, x_n) \right]_{t=0} = 0 \quad (3.37)$$

for any non-negative integer $k(\leq N)$. The solution of Eq.(3.36) with (3.37) is given by

$$h_{n,m}(t; x, x_1, \dots, x_n) = 0 \quad (3.38)$$

for any positive integers n and m . Therefore, when $f(x)$ is a convergent power series of x , we obtain Formula 8 by mathematical induction, because both sides of Eq.(3.33) is linear with respect to the function $f(x)$.

Thus, Theorem IV has been proven. An alternative proof of it is given in Appendix. The third proof is discussed in Sec.VI.

(ii-3) Operator Taylor expansion and shift-hyperoperator $\mathcal{S}_A(B)$

Now we study the Taylor expansion of $f(A + xB)$. First we prove the following general Taylor expansion formula.

Theorem V : When $f(A) \in \mathcal{D}_A$, we have

$$\begin{aligned} f(A + xB) &= \sum_{n=0}^{\infty} x^n \delta_{A \rightarrow B}^n f(A) = \sum_{n=0}^{\infty} \frac{x^n}{n!} d_{A \rightarrow B}^n f(A) \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \frac{d^n f(A)}{dA^n} : \underbrace{B \cdot \dots \cdot B}_n : \end{aligned} \quad (3.39)$$

Equivalently,

$$f(A + xB) = \mathcal{S}_A(xB) f(A), \quad (3.40)$$

where the shift-hyperoperator $\mathcal{S}_A(B)$ is given by

$$\mathcal{S}_A(B) \equiv \sum_{n=0}^{\infty} \frac{1}{n!} (d_{A \rightarrow B})^n = e^{d_{A \rightarrow B}}. \quad (3.41)$$

The proof of this theorem is given as follows. From Eqs.(2.8) and (2.17), we have

$$\delta_{A \rightarrow B}^n A^m = \{A^{m-n} B^n\}_{\text{sym}} = \frac{1}{n!} \left[\frac{d^n}{dx^n} (A + xB)^m \right]_{x=0} \quad (3.42)$$

for $m \geq n$, and we have $\delta_{A \rightarrow B}^n A^m = 0$ for $m < n$. Therefore, we obtain

$$\delta_{A \rightarrow B}^n f(A) = \frac{1}{n!} \left[\frac{d^n}{dx^n} f(A + xB) \right]_{x=0} \quad (3.43)$$

for any positive integer n , when $f(A) \in \mathcal{D}_A$. This yields Theorem V.

In particular, if we put $B = dA$, we obtain the following result.

Theorem VI : *When $f(A) \in \mathcal{D}_A$, we have*

$$f(A + xdA) = f(A) + \sum_{n=1}^{\infty} \frac{x^n}{n!} d^n f(A) = e^{xd} f(A) \quad (3.44)$$

with the differential hyperoperator d defined by (3.23) , namely

$$d \equiv d_{A \rightarrow dA}. \quad (3.45)$$

IV. Multivariate Quantum Analysis

In this section, we formulate multivariate quantum analysis, in which we consider a set of noncommuting power series $\{f(A_1, \dots, A_q)\} \equiv \{f(\{A_k\})\}$. This domain is denoted by $\mathcal{D}_{\{A_k\}}$, namely $f(\{A_k\}) \in \mathcal{D}_{\{A_k\}}$. If we start from a complex number function $f(\{x_k\})$, it is a problem how to define the operator function $f(\{A_k\})$, as is well known in quantum mechanics. Here, we start from the operator function $f(\{A_k\})$ itself which is specified in some appropriate procedures such as normal ordering.

A definition of the partial differential $d_j f(\{A_k\})$ corresponding to Eq.(1.1) is given by

$$d_j f = \lim_{h \rightarrow 0} \frac{f(A_1, \dots, A_j + h d A_j, \dots, A_q) - f(\{A_k\})}{h}. \quad (4.1)$$

Norm convergence of Eq.(4.1) can be discussed in a Banach space and strong convergence is appropriate for unbounded operators. An algebraic partial differential corresponding to Eq.(1.2) is given by

$$d_j f(\{A_k\}) = [H_j, f(\{A_k\})] \quad (4.2)$$

with some auxiliary operators $\{H_j\}$. Both satisfy the Leibniz rule. In the present paper, we study general properties of multivariate quantum derivatives which are invariant for any choice of definitions of differentials. This invariance can be easily proved by extending the procedure shown in IIIA. Namely we have $d_j = \delta_{B_j dA_j; A_j}$ with $B_j = -\delta_{A_j}^{-1}$. The total differential df is defined by

$$df = \sum_{j=1}^q d_j f = (\sum_j d_j) f, \quad (4.3)$$

when $f \in \mathcal{D}_{\{A_k\}}$. The n th differential $d^n f$ is also defined by

$$d^n f = (\sum_j d_j)^n f. \quad (4.4)$$

Clearly, $\{d_j\}$ commute with each other, namely $d_j d_k = d_k d_j$, in the domain $\mathcal{D}_{\{A_k\}}$. One of the key points in the multivariate quantum analysis is to express $d^n f$ in the form

$$d^n f = n! \sum_{j_1, \dots, j_n} f_{j_1, \dots, j_n}^{(n)} : dA_{j_1} \cdots dA_{j_n}. \quad (4.5)$$

Then, we study how to calculate the hyperoperators $\{f_{j_1, \dots, j_n}^{(n)}\}$ in Eq.(4.5).

(i) Ordered differential hyperoperator

In order to study $f_{j_1, \dots, j_n}^{(n)}$, we introduce here an ordered differential hyperoperator d_{j_1, j_2, \dots, j_n} as follows :

$$d_{j_1, j_2, \dots, j_n} = (d_{j_1} d_{j_2} \cdots d_{j_n})_{\text{ordered}}, \quad (4.6)$$

which means $d_{j_1, j_2, \dots, j_n} f(\{A_k\})$ is given by those terms (found via the Leibniz rule) of $d_{j_1} d_{j_2} \cdots d_{j_n} f(\{A_k\})$ in which the differentials appear in the order $dA_{j_1} dA_{j_2} \cdots dA_{j_n}$.

For example, we consider an operator function $f(A, B) = ABA^2$. Then we have

$$\begin{aligned} d_{A,B}f &= (dA)(dB)A^2, \quad d_{B,A}f = A(dB)(dA^2) = A(dB)[(dA)A + AdA], \\ d_{A,A}f &= (dA)BdA^2, \quad d_{B,B}f = 0. \end{aligned} \quad (4.7)$$

Thus, using this ordered differential, we obtain the following formula.

Formula 9 : In the domain $\mathcal{D}_{\{A_k\}}$, we have

$$d_{j_1} \cdots d_{j_n} = \sum_P d_{j_1, \dots, j_n}. \quad (4.8)$$

Here, Σ_P denotes the summation all over the permutations of (j_1, \dots, j_n) .

The proof will be self-evident. In particular, we have the following formulas.

Formula 10 : In the domain $\mathcal{D}_{\{A_k\}}$, we have $d_j d_k = d_{j,k} + d_{k,j}$ and

$$d_j^n = n! \underbrace{d_{j, \dots, j}}_n. \quad (4.9)$$

Formula 11 : In the domain $\mathcal{D}_{\{A_k\}}$, we have

$$d^n f = n! \sum_{j_1, \dots, j_n} d_{j_1, \dots, j_n} f \quad (4.10)$$

for any positive integer n .

Under these preparations, we find a procedure to calculate $\{f_{j_1, \dots, j_n}^{(n)}\}$ in Eq.(4.5). In principle, they are obtained through the following relation :

$$f_{j_1, \dots, j_n}^{(n)} : dA_{j_1} \cdots dA_{j_n} = d_{j_1, \dots, j_n} f. \quad (4.11)$$

Here, $d_{j_1, \dots, j_n} f$ is expressed in the form

$$d_{j_1, \dots, j_n} f = \sum_k f_{k,0}(dA_{j_1}) f_{k,1}(dA_{j_2}) f_{k,2} \cdots (dA_{j_n}) f_{k,n} \quad (4.12)$$

with some appropriate operators $\{f_{k,j}\}$. In order to find $\{f_{j_1, \dots, j_n}^{(n)}\}$ explicitly, we have to rearrange Eq.(4.12) in the form of the left-hand side of Eq.(4.11). For this purpose, the following rearrangement formula¹³ will be useful.

Formula 12 (Rearrangement formula) : Any product $Q_1 f_1 Q_2 f_2 \cdots Q_n f_n$ can be rearranged in the form

$$Q_1 f_1 \cdots Q_n f_n = \sum_{j=1}^{n+1} f_1 f_2 \cdots f_{j-1} \sum_{\pi} \partial_{\pi(j, j_1)} \partial_{\pi(j_1, j_2)} \cdots \partial_{\pi(j_k, n+1)} : Q_1 \cdots Q_n \quad (4.13)$$

with $f_0 = 1$. Here, \sum_{π} denotes the summation all over the ways of the following division of the set of natural numbers $(j, j+1, \dots, n-1, n)$:

$$(j, j+1, \dots, n-1, n) = \pi(j, j_1) \pi(j_1, j_2) \cdots \pi(j_k, n+1), \quad (4.14)$$

and

$$\pi(j, k) = (j, j+1, \dots, k-1) \quad (4.15)$$

with $j < j_1 < \cdots < j_k \leq n$. Furthermore, the hyperoperator $\partial_{\pi(j, k)}$ is defined by

$$\partial_{\pi(j, k)} = -\delta_{f_j f_{j+1} \cdots f_{k-1}; Q_j}, \quad (4.16)$$

using the partial inner derivation $\delta_{f; Q_j} \equiv \delta_{f; j}$ which operates only on Q_j in (4.13).

The proof of Formula 12 is easily given by mathematical induction.

It will be instructive to give here some examples :

$$\begin{aligned} Q_1 f_1 &= (f_1 - \delta_{f_1}) : Q_1, \\ Q_1 f_1 Q_2 f_2 &= (f_1 f_2 - f_1 \delta_{f_2; 2} - \delta_{f_1 f_2; 1} + \delta_{f_1; 1} \delta_{f_2; 2}) : Q_1 \cdot Q_2, \\ Q_1 f_1 Q_2 f_2 Q_3 f_3 &= (f_1 f_2 f_3 - f_1 f_2 \delta_{f_3; 3} + f_1 \delta_{f_2; 2} \delta_{f_3; 3} - f_1 \delta_{f_2 f_3; 2} \\ &\quad - \delta_{f_1 f_2 f_3; 1} + \delta_{f_1; 1} \delta_{f_2 f_3; 2} + \delta_{f_1 f_2; 1} \delta_{f_3; 3} - \delta_{f_1; 1} \delta_{f_2; 2} \delta_{f_3; 3}) : Q_1 \cdot Q_2 \cdot Q_3. \end{aligned} \quad (4.17)$$

(ii) Partial derivative and multivariate operator Taylor expansion

It will be convenient to define the following partial quantum derivative

$$\frac{\partial^n f}{\partial A_{j_n} \cdots \partial A_{j_1}} \equiv n! f_{j_1, \dots, j_n}^{(n)}, \quad (4.18)$$

using the hyperoperators $\{f_{j_1, \dots, j_n}^{(n)}\}$ determined through the relation (4.11). Then, we obtain the following theorem.

Theorem VII : *When $f(\{A_k\}) \in \mathcal{D}_{\{A_k\}}$, we have*

$$\begin{aligned} f(\{A_j + x_j dA_j\}) &= \sum_{n=0}^{\infty} \sum_{j_1, \dots, j_n} x_{j_1} \cdots x_{j_n} d_{j_1, \dots, j_n} f \\ &= \sum_{n=0}^{\infty} \sum_{j_1, \dots, j_n} x_{j_1} \cdots x_{j_n} f_{j_1, \dots, j_n}^{(n)} : dA_{j_1} \cdots dA_{j_n} \\ &= \sum_{n=0}^{\infty} \sum_{j_1, \dots, j_n} \frac{x_{j_1} \cdots x_{j_n}}{n!} \frac{\partial^n f}{\partial A_{j_n} \cdots \partial A_{j_1}} : dA_{j_1} \cdots dA_{j_n}. \end{aligned} \quad (4.19)$$

Equivalently, we have

$$f(\{A_j + x_j B_j\}) = \exp \left(\sum_{j=1}^q x_j d_{A_j \rightarrow B_j} \right) f(\{A_j\}). \quad (4.20)$$

In particular,

$$f(\{A_j + x dA_j\}) = e^{xd} f(\{A_j\}) = \sum_{n=0}^{\infty} \frac{x^n}{n!} d^n f(\{A_j\}) \quad (4.21)$$

with $d = \sum_j d_j$.

V. Auxiliary Operator Method

It will be convenient to introduce the auxiliary operators $\{H_j\}$ satisfying the following conditions:

$$[H_j, H_k] = 0, [H_j, A_k] = 0, [H_j, [H_k, A_k]] = 0, \text{ for } j \neq k \quad (5.1)$$

and

$$[H_j, [H_j, A_j]] = 0. \quad (5.2)$$

Using these auxiliary operators $\{H_j\}$, we introduce the following partial differential

$$d_j f \equiv [H_j, f] \equiv \delta_{H_j} f. \quad (5.3)$$

In particular, we have

$$dA_j = d_j A_j = [H_j, A_j], \quad (5.4)$$

and

$$d^2 A_j = 0 \quad \text{and} \quad d_j(dA_k) = 0. \quad (5.5)$$

The total differential df is given by

$$df \equiv \sum_j [H_j, f] = (\sum_j d_j) f. \quad (5.6)$$

One of the merits of this auxiliary operator method is that we can easily obtain the operator Taylor expansion as follows :

$$\begin{aligned} \exp(\sum_j x_j \delta_{H_j}) f(\{A_j\}) &= f(\{\exp(x_j \delta_{H_j}) A_j\}) \\ &= f(\{A_j + x_j \delta_{H_j} A_j\}) = f(\{A_j + x_j dA_j\}), \end{aligned} \quad (5.7)$$

using Eqs. (5.2) and (5.4). That is, we have

$$f(\{A_j + x_j dA_j\}) = \exp(\sum_j x_j \delta_{H_j}) f(\{A_j\}) = e^{\sum_j x_j d_j} f(\{A_j\}), \quad (5.8)$$

using the relation (5.3).

VI. Some General Remarks and Applications to Exponential Product Formulas

It will be instructive to remark that when the operator A depends on a parameter t , namely $A = A(t)$, we have¹⁰

$$\frac{df(A(t))}{dt} = \frac{df(A(t))}{dA(t)} \cdot \frac{dA(t)}{dt}. \quad (6.1)$$

This formula insures again the invariance of the derivative $df(A(t))/dA$, because $df(A(t))/dt$ and $dA(t)/dt$ do not depend on the choice of the differential $df(A(t))$.

Furthermore, we have

$$\frac{df(g(A))}{dA} = \frac{df(g(A))}{dg(A)} \cdot \frac{dg(A)}{dA}, \quad (6.2)$$

because

$$df(g(A)) = \frac{df(g(A))}{dg(A)} \cdot dg(A) = \frac{df(g(A))}{dg(A)} \cdot \frac{dg(A)}{dA} : dA. \quad (6.3)$$

It will be also interesting to note the derivative of hyperoperators. The first differential of a hyperoperator $f(\delta_A)$ is given by¹⁰

$$d(f(\delta_A)dA) = \int_0^1 dt[f^{(1)}(t\delta_1 + \delta_2) - f^{(1)}(\delta_1 + t\delta_2)] : (dA)^2. \quad (6.4)$$

In general, we have

$$\begin{aligned} d[f(A; \delta_1, \dots, \delta_n) : (dA)^n] &= \int_0^1 dt f^{(1)}(A - t\delta_1; \delta_2, \dots, \delta_{n+1}) : (dA)^{n+1} \\ &+ \sum_{k=1}^n \left\{ \int_0^1 dt_k \left[f^{(1,k)}(A; \delta_1, \dots, \delta_{k-1}, t_k\delta_k + \delta_{k+1}, \delta_{k+2}, \dots, \delta_{n+1}) \right. \right. \\ &\left. \left. - f^{(1,k)}(A; \delta_1, \dots, \delta_{k-1}, \delta_k + t_k\delta_{k+1}, \delta_{k+2}, \dots, \delta_{n+1}) \right] : (dA)^{n+1} \right\}. \end{aligned} \quad (6.5)$$

Here, $f^{(1)}(x; x_1 \dots, x_n)$ denotes the first derivative of $f(x; x_1 \dots, x_n)$ with respect to x and $f^{(1,k)}(x; x_1 \dots, x_k, \dots, x_n)$ denotes the first derivative of f with respect to x_k . Note that A and $\{\delta_k\}$ commute with each other. These formulas will be also useful in proving Theorem IV. In fact, we obtain

$$\begin{aligned} d^2 f(A) &= d(df(A)) \\ &= d\left(\int_0^1 dt f^{(1)}(A - t\delta_A)dA\right) \\ &= \int_0^1 dt d(f^{(1)}(A - t\delta_A) \cdot dA) \\ &= \int_0^1 dt_1 \int_0^1 dt_2 f^{(2)}(A - t_1\delta_1 - t_2\delta_2) : (dA)^2 \\ &+ \int_0^1 dt \int_0^1 (-s)ds [f^{(2)}(A - s(t\delta_1 + \delta_2)) - f^{(2)}(A - s(\delta_1 + t\delta_2))] : (dA)^2 \\ &= 2 \int_0^1 dt_1 \int_0^{t_1} dt_2 f^{(2)}(A - t_1\delta_1 - t_2\delta_2) : (dA)^2. \end{aligned} \quad (6.6)$$

Similarly we can derive Theorem IV using the above formula (6.5).

There are many applications of quantum analysis to exponential product formulas^{11–15} such as the Baker-Campbell-Hausdorff formula.

For example, if we put

$$e^{A_1(x)} e^{A_2(x)} \cdots e^{A_r(x)} = e^{\Phi(x)}, \quad (6.7)$$

the operator $\Phi(x)$ is shown to satisfy the operator equation¹⁴

$$\frac{d\Phi(x)}{dx} = \Delta^{-1}(\Phi(x)) \sum_{j=1}^r \exp(\delta_{A_1(x)}) \cdots \exp(\delta_{A_{j-1}(x)}) \Delta(A_j(x)) \frac{dA_j(x)}{dx} \quad (6.8)$$

using the quantum derivative of e^A :

$$\frac{de^A}{dA} = \frac{e^A - e^{A-\delta_A}}{\delta_A} = e^A \Delta(-A); \Delta(A) = \frac{e^{\delta_A} - 1}{\delta_A}. \quad (6.9)$$

The solution of Eq.(6.8) is given by

$$\begin{aligned} \Phi(x) &= \sum_{j=1}^r \int_0^x \frac{\log[\exp(\delta_{A_1(t)}) \cdots \exp(\delta_{A_r(t)})]}{\exp(\delta_{A_1(t)}) \cdots \exp(\delta_{A_r(t)}) - 1} \\ &\quad \times \exp(\delta_{A_1(t)}) \cdots \exp(\delta_{A_{j-1}(t)}) \Delta(A_j(t)) \frac{dA_j(t)}{dt} dt + \Phi(0). \end{aligned} \quad (6.10)$$

This is a generalized BCH formula.

In particular, we have

$$\log(e^A e^B e^A) = \int_0^1 \left(\frac{e^{t\delta_A} e^{\delta_B} e^{t\delta_A} + 1}{e^{t\delta_A} e^{\delta_B} e^{t\delta_A} - 1} \log(e^{t\delta_A} e^{\delta_B} e^{t\delta_A}) \right) A dt + B. \quad (6.11)$$

Recursively we have

$$\log(e^{A_1} \cdots e^{A_r}) = \int_0^1 dt \frac{\log E_r(t)}{E_r(t) - 1} (A_1 + E_r(t) A_r) + \Phi_{2,r-1}, \quad (6.12)$$

where $\Phi_{2,r-1} = \log(e^{A_2} \cdots e^{A_{r-1}})$, and

$$E_r(t) = \exp(t\delta_{A_1}) \exp(\delta_{A_2}) \cdots \exp(\delta_{A_{r-1}}) \exp(t\delta_{A_r}). \quad (6.13)$$

The feature of these formulas is that $\Phi(x)$ and $\log(e^{A_1} \cdots e^{A_r})$ are expressed only in terms of linear combinations of $\{A_j\}$ and their commutators.

These formulas will be useful in studying higher-order decomposition formulas¹⁷.

VII. Summary and Discussion

In the present paper, we have unified an analytic formulation of quantum analysis based on the Gâteaux differential and an algebraic formulation of quantum analysis based on commutators, by introducing the two hyperoperators $\delta_{A \rightarrow B} \equiv -\delta_A^{-1} \delta_B$ and $d_{A \rightarrow B} \equiv \delta_{(-\delta_A^{-1} B); A}$. This general theory of quantum analysis gives a proof of the invariance of quantum derivatives for any choice of the definitions of differentials in the domain \mathcal{D}_A . This domain can be easily extended¹² to the region $\tilde{\mathcal{D}}_A$ which is a set of convergent Laurent series of the operator A in a Banach space. Multivariate quantum derivatives have also been formulated using the rearrangement formula.

The present general formulation will be used effectively in studying quantum fluctuations in condensed matter physics and it will be also useful in mathematical physics. The present quantum analysis can also be extended to an infinite number of variables¹⁴. The quantum analysis has been also used¹⁵ in extending Kubo's linear response theory¹⁸ and Zubarev's theory of statistical operator¹⁹ to more general nonlinear situations¹¹. The invariant property of quantum derivatives derived in Sec.III is closely related¹⁵ to the general feature of the fluctuation-dissipation theorem¹⁸⁻²¹. General quantum correlation identities are also derived¹⁵ using the quantum analysis. For the convergence of unbounded operators, see the second paper of Ref.17.

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Appendix : Alternative Proof of Theorem IV

First we study the case $f(A) = A^m$ for a positive integer m . The n th differential $d^n A^m$ is expressed in the form

$$\begin{aligned}
d^n A^m &= d_{A \rightarrow dA}^n A^m = n! \delta_{A \rightarrow dA}^n A^m = n! \{A^{m-n} (dA)^n\}_{\text{sym}} \\
&= n! \sum_{k_j \geq 0, \sum k_j = m-n} A^{k_0} (dA)^{k_1} \cdots A^{k_{n-1}} (dA)^{k_n} \\
&= n! \sum_{k_j \geq 0, \sum k_j = m-n} A^{k_0} (A - \delta_1)^{k_1} \\
&\quad \cdots (A - \delta_1 - \cdots - \delta_n)^{k_n} : dA \cdots dA,
\end{aligned} \tag{A.1}$$

for $m \geq n$ and $d^n A^m = 0$ for $n > m$, using Theorem III, Formula 5, Formula 2, the definition of the symmetrized product, Eq.(2.7), and the following formula¹⁰.

Formula A : For any operator Q , we have

$$Qf(A) = f(A - \delta_A)Q \tag{A.2}$$

when $f(A) \in \mathcal{D}_A$.

This yields Lemma 2. Now, we prove the following lemma.

Lemma A : When $f(A) = A^m$ with a positive integer $m(\geq n)$, the formula (3.28) holds. That is, we have

$$\sum_{k_j \geq 0, \sum k_j = m-n} A^{k_0} (A - \delta_1)^{k_1} \cdots (A - \delta_1 - \cdots - \delta_n)^{k_n} = F_n(A; \delta_1 \cdots, \delta_n), \tag{A.3}$$

where

$$F_n(A; \delta_1, \dots, \delta_n) \equiv \frac{m!}{(m-n)!} \int_0^1 dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n (A - \sum_{j=1}^n t_j \delta_j)^{m-n}. \quad (\text{A.4})$$

This lemma can be proved by mathematical induction as follows. We assume that Eq.(A.3) holds in the case of $F_{n-1}(A; \delta_1, \dots, \delta_{n-1})$. Then, we have

$$\begin{aligned} F_n(A; \delta_1, \dots, \delta_n) &= \frac{m!}{(m-n+1)!} \int_0^1 dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-2}} dt_{n-1} \\ &\quad \times \frac{1}{\delta_n} \left[(A - \sum_{j=1}^{n-1} t_j \delta_j)^{m-n+1} - \{A - \sum_{j=1}^{n-2} t_j \delta_j - t_{n-1}(\delta_{n-1} + \delta_n)\}^{m-n+1} \right] \\ &= \frac{1}{\delta_n} \sum_{k_j \geq 0, \sum k_j = m-n+1} A^{k_0} (A - \delta_1)^{k_1} \cdots (A - \delta_1 - \cdots - \delta_{n-2})^{k_{n-2}} \\ &\quad \times \{ (A - \delta_1 - \cdots - \delta_{n-1})^{k_{n-1}} - (A - \delta_1 - \cdots - \delta_n)^{k_{n-1}} \} \end{aligned} \quad (\text{A.5})$$

under the assumption that Eq.(A.3) holds for $F_{n-1}(A; \delta_1, \dots, \delta_{n-1})$. Then, the above expression (A.5) can be rearranged as

$$\begin{aligned} F_n(A; \delta_1, \dots, \delta_n) &= \sum_{k_j \geq 0, \sum k_j = m-n+1} A^{k_0} (A - \delta_1)^{k_1} \cdots (A - \delta_1 - \cdots - \delta_{n-2})^{k_{n-2}} \\ &\quad \times \sum_{k'_{n-1} \geq 0, k'_n \geq 0, k'_{n-1} + k'_n = k_{n-1} - 1} (A - \delta_1 - \cdots - \delta_{n-1})^{k'_{n-1}} \cdot (A - \delta_1 - \cdots - \delta_n)^{k'_n} \\ &= \sum_{k_j \geq 0, \sum k_j = m-n} A^{k_0} (A - \delta_1)^{k_1} \cdots (A - \delta_1 - \cdots - \delta_n)^{k_n}, \end{aligned} \quad (\text{A.6})$$

by noting that $k_0 + k_1 + \cdots + k_{n-2} + k'_{n-1} + k'_n = k_0 + k_1 + \cdots + k_{n-1} - 1 = (m-n+1) - 1 = m-n$. Thus, we arrive at Lemma A. Any operator $f(A) \in \mathcal{D}_A$ is expressed as a power series of $\{A^m\}$. Then, Lemma A yields Theorem IV.

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